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ECON 100A Section 111, 112

Lagrange Multiplier (Version 3.1)

The method of Lagrange Multiplier is a very general method of solving optimization problems. This handout will guide you through its usage.

I. Setup the Lagrangian

The very first step in using Lagrange Multiplier is to setup the Lagrangian formula. Following the lecture slides I shall denote the Lagrangian as Φ, but it is very commonly denoted as *L*. Φ has the following general setup,

$$\boldsymbol{\Phi} = \underbrace{\boxed{\phantom{\text{The function to optimize}}}}_{\textit{The function to optimize}} + \boldsymbol{\lambda} \cdot \underbrace{\boxed{\phantom{\text{An equality constraint}}}}_{\textit{An equality constraint}}$$

What should we put inside the boxes? Here is a list for some of the common topics,

Topic	The Function to Optimize	Equality Constraint
Utility Maximization (UM)	$U(x, y)$	$I - P_x x - P_y y$
Product Maximization (PM)	$F(K, L)$	$\overline{C} - rK - wL$
Cost Minimization (CM)	$rK + wL$	$\overline{F} - F(K, L)$

Note the form of the equality constraints; we get them by moving all the terms to one side.

$$P_x x + P_y y = I \quad \rightarrow \quad I - P_x x - P_y y = 0 \quad \rightarrow \quad I - P_x x - P_y y$$

$$rK + wL = \overline{C} \quad \rightarrow \quad \overline{C} - rK - wL = 0 \quad \rightarrow \quad \overline{C} - rK - wL$$

$$F(K, L) = \overline{F} \quad \rightarrow \quad \overline{F} - F(K, L) = 0 \quad \rightarrow \quad \overline{F} - F(K, L)$$

The optimization solution does not depend on which side we move the terms to; the only thing that side affects is the sign of λ.

II. First Order Conditions

The second step is to take the first order conditions (FOC) with respect to the applicable variables and λ. FOC are found by equating partial derivatives of Φ with respect to each variable to 0.

Topic	Variables	FOC
UM	x, y	$\frac{\partial \Phi}{\partial x}=0$ $\frac{\partial \Phi}{\partial y}=0$ $\frac{\partial \Phi}{\partial \lambda}=0$
PM	K, L	$\frac{\partial \Phi}{\partial K}=0$ $\frac{\partial \Phi}{\partial L}=0$ $\frac{\partial \Phi}{\partial \lambda}=0$
CM	K, L	$\frac{\partial \Phi}{\partial K}=0$ $\frac{\partial \Phi}{\partial L}=0$ $\frac{\partial \Phi}{\partial \lambda}=0$

Refer back to the setup of Φ we have

Topic	FOC		
UM	$\frac{\partial \Phi}{\partial x}=0$ $\frac{\partial \Phi}{\partial y}=0$ $\frac{\partial \Phi}{\partial \lambda}=0$	$\rightarrow$ $\rightarrow$ $\rightarrow$	$\frac{\partial U}{\partial x}-\lambda P_x=0$ $\frac{\partial U}{\partial y}-\lambda P_y=0$ $I-P_x x-P_y y=0$
PM	$\frac{\partial \Phi}{\partial K}=0$ $\frac{\partial \Phi}{\partial L}=0$ $\frac{\partial \Phi}{\partial \lambda}=0$	$\rightarrow$ $\rightarrow$ $\rightarrow$	$\frac{\partial F}{\partial K}-\lambda r=0$ $\frac{\partial F}{\partial L}-\lambda w=0$ $\overline{C}-rK-wL=0$
CM	$\frac{\partial \Phi}{\partial K}=0$ $\frac{\partial \Phi}{\partial L}=0$ $\frac{\partial \Phi}{\partial \lambda}=0$	$\rightarrow$ $\rightarrow$ $\rightarrow$	$r-\lambda \frac{\partial F}{\partial K}=0$ $w-\lambda \frac{\partial F}{\partial L}=0$ $\overline{F}-F(K,L)=0$

III. Solving the First Order Conditions

Recall what we have learnt in consumer choice theory—we solve for the optimal allocation with EMP and budget constraint. This is exactly what the first order conditions give us—for any optimization problem the optimal interior allocation is obtained by solving the set of first order conditions.

<u>FOC 1 & 2 – Equal-Marginal Principle</u>

Combining the first two FOC we get EMP for all cases:

Topic	FOC 1 & 2				
UM	$\frac{\partial U}{\partial x} - \lambda P_x = 0$ $\frac{\partial U}{\partial y} - \lambda P_y = 0$	$\rightarrow$	$\frac{\partial U}{\partial x} \Big/ P_x = \lambda$ $\frac{\partial U}{\partial y} \Big/ P_y = \lambda$	$\rightarrow$	$\frac{\left(\frac{\partial U}{\partial x}\right)}{P_x} = \frac{\left(\frac{\partial U}{\partial y}\right)}{P_y}$
PM	$\frac{\partial F}{\partial K} - \lambda r = 0$ $\frac{\partial F}{\partial L} - \lambda w = 0$	$\rightarrow$	$\frac{\partial F}{\partial K} \Big/ r = \lambda$ $\frac{\partial F}{\partial L} \Big/ w = \lambda$	$\rightarrow$	$\frac{\left(\frac{\partial F}{\partial K}\right)}{r} = \frac{\left(\frac{\partial F}{\partial L}\right)}{w}$
CM	$r - \lambda \frac{\partial F}{\partial K} = 0$ $w - \lambda \frac{\partial F}{\partial L} = 0$	$\rightarrow$	$r \Big/ \frac{\partial F}{\partial K} = \lambda$ $w \Big/ \frac{\partial F}{\partial L} = \lambda$	$\rightarrow$	$\frac{\left(\frac{\partial F}{\partial K}\right)}{r} = \frac{\left(\frac{\partial F}{\partial L}\right)}{w}$

Solving EMP gives us the optimal proportion of the variables (e.g. $x = 4y$ or $K = L/4$); substituting this optimal proportion into FOC 3 we can get the optimal values for each of the variables.

<u>FOC 3 – Equality Constraint</u>

FOC with respect to λ always gives us back the equality constraint:

Topic	FOC		
UM	$\frac{\partial \Phi}{\partial \lambda} = 0$	$\rightarrow$	$I - P_x x - P_y y = 0$
PM	$\frac{\partial \Phi}{\partial \lambda} = 0$	$\rightarrow$	$\overline{C} - rK - wL = 0$
CM	$\frac{\partial \Phi}{\partial \lambda} = 0$	$\rightarrow$	$\overline{F} - F(K, L) = 0$

IV. Interpretation of λ

λ represents the marginal change in the function to optimize with respect to a relaxation of the equality constraint. What does that mean? The following table tells:

Topic	Equality Constraint	So relaxation means…
Utility Maximization	$I - P_x x - P_y y$	An increase in I
Product Maximization	$\overline{C} - rK - wL$	An increase in $\overline{C}$
Cost Minimization	$\overline{F} - F(K, L)$	An increase in $\overline{F}$

So for instance in utility maximization, λ represents the marginal change in utility with an increase in income—in other words *the marginal utility of income*. The following table summarizes,

Topic	λ represents…
Utility Maximization	Marginal utility of income
Product Maximization	Marginal product of expenditure
Cost Minimization	Marginal cost of production

λ is often called the *shadow price*. Note that λ's sign definitely matters here; if we use $P_x x + P_y y - I$ as the equality constraint instead of $I - P_x x - P_y y$ we get λ of an opposite sign; in such case $-\lambda$ is the marginal utility of income.

For easy reference, remember the value of λ is

Topic	Value of λ
UM	$\lambda = \frac{\left(\frac{\partial U}{\partial x}\right)}{P_x} = \frac{\left(\frac{\partial U}{\partial y}\right)}{P_y}$
PM	$\lambda = \frac{\left(\frac{\partial F}{\partial K}\right)}{r} = \frac{\left(\frac{\partial F}{\partial L}\right)}{w}$
CM	$\lambda = \frac{r}{\left(\frac{\partial F}{\partial K}\right)} = \frac{w}{\left(\frac{\partial F}{\partial L}\right)}$

Extra Material

Multiple Constraints and More Variables

The method of Lagrange Multiplier can be generalized to multiple equality constraints and more variables. The only difference from what we have learnt is each constraint gets its own λ. So if $f(x_1,...,x_N)$ is the function to be optimized and

$g_j(x_1,...,x_N)$, $j = 1,\ldots,J$ are equality constraints, the Lagrangian is

$$\Phi = f(x_1...,x_N) + \sum_{j=1}^{J} \lambda_j g_j(x_1,...,x_N)$$

Second-Order Condition (SOC)

In general we need to check SOC in order to make sure whether the solution we get from solving FOC is a maximum, minimum or neither; this involves computing a matrix of second derivatives called the *Hessian Matrix*. You will not be asked to do so in this class.

Proof: The interpretation of λ

I will work with utility maximization; the proof is identical for the other two cases. We start with the definition of marginal utility of income—the partial derivative of the utility function with respect to income. Since the utility function depends only on *x* and *y*, any effect of income must exhibit through *x* and *y*; so

$$\frac{\partial U}{\partial I} = \frac{\partial U}{\partial x}\frac{\partial x}{\partial I} + \frac{\partial U}{\partial y}\frac{\partial y}{\partial I} \tag{1}$$

By FOC 1, $\frac{\partial U}{\partial x} = \lambda P_x$; by FOC 2, $\frac{\partial U}{\partial x} y = \lambda P_y$. Substituting these into (1) we have

$$\begin{aligned}\frac{\partial U}{\partial I} &= \lambda P_x \frac{\partial x}{\partial I} + \lambda P_y \frac{\partial x}{\partial I} \\ &= \lambda\left(P_x \frac{\partial x}{\partial I} + P_y \frac{\partial x}{\partial I}\right)\end{aligned} \tag{2}$$

Now consider the equality (budget) constraint,

$$P_x x + P_y y = I$$

Differentiate both sides with respect to *I* gives

$$P_x \frac{\partial x}{\partial I} + P_y \frac{\partial y}{\partial I} = 1 \tag{3}$$

Substitute this to (2) we have

$$\frac{\partial U}{\partial I} = \lambda(1) = \lambda \tag{4}$$