University of California, Berkeley — Fall 2006
ECON 100A Section 111, 112

Maximizing Expected Utility with Calculus

The method I am going to introduce works for quite specific cases; please verify that the assumption is true before using it.

I. Lottery Tickets

Setting

Utility function:	$U(w)$
Initial Wealth:	$\overline{w}$
Number of tickets:	S, F
Prices for lottery tickets S and F:	p_S, p_F
Payoffs for ticket S and F:	x_S, x_F
Probability of S winning:	$\Pr_S$

Then expected utility is

$$E[U(w)] = \Pr_S U(\overline{w} - p_S \cdot S - p_F \cdot F + x_S \cdot S + 0 \cdot F) + (1 - \Pr_S) U(\overline{w} - p_S \cdot S - p_F \cdot F + 0 \cdot S + x_F \cdot F)$$

The optimization problem is

$$\max_{S,F} \{E[U(w)]\}$$

Assumption: $p_S + p_F = x_S = x_F$

Let $D = S - F$, with the above assumption we have

$$E[U(w)] = \Pr_S U(\overline{w} - p_S \cdot D) + (1 - \Pr_S) U(\overline{w} - p_S \cdot D + x_F \cdot D)$$

So the optimization problem is

$$\max_{D} \{\Pr_S U(\overline{w} - p_S \cdot D) + (1 - \Pr_S) U(\overline{w} - p_S \cdot D + x_F \cdot D)\}$$

First Order Condition

$$\Pr_S U'(\overline{w} - p_S \cdot D^*)(-p_S) + (1 - \Pr_S) U'(\overline{w} - p_S \cdot D^* + x_F \cdot D^*)(x_F - p_S) = 0$$

Solving FOC gives us the optimal D^*. Since $D = S - F$, any combination of (S, F) that satisfies $S - F = D^*$ is optimal.

Example:

<u>Practice Midterm #23</u>

Utility function:	$\ln(w)$
Initial Wealth:	1800
Prices for lottery tickets S and F:	$p_S = 1, p_F = 9$
Payoffs for ticket S and F:	$x_S = x_F = 10$
Probability of S winning:	$\frac{1}{2}$

So the optimization problem is

$$\max_D \left\{ \tfrac{1}{2} \cdot \ln(1800 - 1 \cdot D) + \tfrac{1}{2} \cdot \ln(1800 - 1 \cdot D + 10 \cdot D) \right\}$$

FOC:

$$\frac{1}{2} \cdot \frac{1}{1800 - D^*} \cdot (-1) + \frac{1}{2} \cdot \frac{1}{1800 - D^* + 10D^*} \cdot (10 - 1) = 0$$

$$\frac{1}{2} \cdot \frac{1}{1800 - D^* + 10D^*} \cdot 9 = \frac{1}{2} \cdot \frac{1}{1800 - D^*}$$

$$9(1800 - 1 \cdot D^*) = 1800 + 9D^*$$

$$9 \cdot 1800 - 9D^* = 1800 + 9D^*$$

$$D^* = 800$$

So any (S, F) with $S - F = 800$ is a solution. Within the choices only $(S, F) = (900,100)$ satisfies this so it (d) is the correct answer.

II. Insurance

<u>Setting</u>

Utility function:	$U(w)$
Initial Wealth:	$\overline{w}$
Wealth if the bad event happens:	$\underline{w}$
Amount of insurance:	S
Price of each dollar of insurance:	p
Probability of bad event:	$\Pr_B$

Expected utility is

$$E[U(w)] = \Pr_B U(\overline{w} - p \cdot S) + (1 - \Pr_B) U(\underline{w} - p \cdot S)$$

The optimization problem is

$$\max_{S}\left\{\Pr_B U(\overline{w}-p\cdot S)+(1-\Pr_B)U(\underline{w}-p\cdot S+S)\right\}$$

First Order Condition

$$\Pr_B U'(\overline{w}-p\cdot S^*)(-p)+(1-\Pr_B)U'(\underline{w}-p\cdot S^*+S^*)(1-p)=0$$

Solving FOC gives us the optimal K^*.

Example:

<u>Practice Midterm #26</u>

Utility function:	$\ln(w)$
Initial Wealth:	200,000
Wealth if the bad event happens:	100,000
Amount of insurance:	k
Price of each dollar of insurance:	0.6
Probability of bad event:	0.5

So the optimization problem is

$$\max_{k}\left\{0.5\ln(200{,}000-0.6k)+0.5\ln(100{,}000-0.6k+k)\right\}$$

FOC:

$$0.5\cdot\frac{1}{200{,}000-0.6k^*}\cdot(-0.6)+0.5\cdot\frac{1}{100{,}000-0.6k^*+k^*}\cdot(1-0.6)=0$$

$$0.5\cdot\frac{1}{100{,}000-0.6k^*+k^*}\cdot(1-0.6)=0.5\cdot\frac{1}{200{,}000-0.6k^*}\cdot(0.6)$$

$$\frac{4}{6}\left[200{,}000-0.6k^*\right]=100{,}000-0.6k^*+k^*$$

$$k^*=\frac{1}{0.8}\left(\frac{4}{6}200{,}000-100{,}000\right)$$

$$k^*=\frac{2{,}000{,}000}{48}$$

$$\sim 40{,}000$$

k^* is more than \$0 but less than \$50,000, so (c) is the correct answer.